AIAA Journal
Vol. 47, No. 3, March 2009

Efficient Methods for Time-Dependent Fatigue Reliability Analysis

Yongming Liu*
Clarkson University, Potsdam, New York 13699
and
Sankaran Mahadevan
Vanderbilt University, Nashville, Tennessee 37235

DOI: 10.2514/1.34331

Two efficient methods for time-dependent fatigue reliability analysis are proposed in this paper based on a random process representation of material fatigue properties and a nonlinear damage accumulation rule. The first method is developed by matching the first two central moments of the accumulated damage to a well-known probability distribution, thus facilitating a direct analytical solution of the time-dependent fatigue reliability. The second method uses the first-order reliability method to calculate the reliability index based on a time-dependent limit state function. These two methods represent different tradeoffs between accuracy and computational efficiency. The proposed methods include the covariance structure of the stochastic damage accumulation process under variable amplitude loading. A wide range of fatigue data available in the literature is used to validate the proposed methods, covering several different types of metallic and composite materials under different variable amplitude loading.

I. Introduction

THE fatigue process of mechanical components under service loading is stochastic in nature. The prediction of time-dependent fatigue reliability is critical for the design and maintenance planning of many structural components. Despite extensive progress made in the past decades, life prediction and reliability evaluation is still a challenging problem. Two types of probability distributions are often used to characterize the randomness of the fatigue damage accumulation and fatigue life. One is the probabilistic life distribution, that is, the distribution of service time (life) to exceed a critical damage value. The other is the probabilistic damage distribution, that is, the distribution of the amount of damage at any service time. Time-dependent fatigue reliability refers to the latter one (i.e., the probability of damage being less than a critical value at time t). Both simulation-based and simplified approximation methods can be used to estimate the time-dependent reliability. Liu and Mahadevan [1] proposed a Monte Carlo simulation methodology to calculate the probabilistic fatigue life distribution and validated it for various metallic materials. The objective of this study is to develop a simple approximation methodology to calculate the time-dependent fatigue reliability.

The following problems need to be carefully solved to accurately predict the time-dependent fatigue reliability: uncertainty quantification of material properties, uncertainty quantification of applied loading, and an appropriate damage accumulation rule. Different approaches have been proposed to handle these problems. For uncertainty quantification of material properties, two main approaches exist in the literature to represent experimental data under constant amplitude loading. One approach assumes that fatigue lives at different stress levels are independent random variables [2–6]. The other approach assumes that fatigue lives at different stress levels are fully dependent random variables [7–13]. Liu and Mahadevan [1] proposed a stochastic *S-N* curve representation technique to include the actual correlation of fatigue lives across different stress levels. The two approaches using independent or fully dependent assumptions are two special cases of the developed methodology [1].

The damage accumulation rule is another important component in time-dependent fatigue reliability analysis. The linear damage accumulation rule, also known as Miner's rule, is commonly used because of its simplicity. The major deficiency of the linear damage accumulation rule is that it cannot consider the load dependence effect. Nonlinear damage accumulation rules, such as the damage curve approach [14] and the double linear curve approach [15], can consider the load dependence effect but require cycle-by-cycle calculation, which significantly increases the computational cost especially for probabilistic analysis. Liu and Mahadevan [1] proposed a modification of the linear damage accumulation rule to overcome its deficiency while maintaining the simplicity of computational effort. This modified damage accumulation rule is used in this study.

This paper develops two approximate methods to calculate fatigue reliability as a function of time t. The first method matches the first two central moments of the accumulated damage to a well-known distribution facilitating quick analytical calculation of the time-dependent reliability. The second method does not assume the distribution of the accumulated damage and uses the first-order reliability method (FORM) to calculate the reliability index of a time-dependent limit state function. Both methods are initially developed for stationary loading and then extended for nonstationary loading. Following this, the prediction results of the two methods are compared with direct Monte Carlo simulation and found to be very efficient and accurate. Several sets of experimental data under variable amplitude loading are used to validate the proposed methods.

II. Uncertainty Quantification and Damage Accumulation Modeling

A. Uncertainty Quantification of External Loading

Two approaches are commonly used to describe the scatter in the random applied loading. One is in the frequency domain and uses power spectral density methods. The other is in the time domain and uses cycle counting techniques. The major advantages of the frequency domain approach are that it is more efficient and can obtain an analytical solution under some assumptions of the applied loading process, such as the Gaussian process, stationary, and narrowbanded. This of course limits the applicability of the frequency domain approach [16,17]. Also, most frequency domain analyses assume a

---

Received 29 August 2007; accepted for publication 30 September 2008. Copyright © 2008 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved. Copies of this paper may be made for personal or internal use, on condition that the copier pay the $10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 0001-1452/09 $10.00 in correspondence with the CCC.

*yliu@clarkson.edu (Corresponding Author).

linear fatigue damage accumulation rule [18–20], due to its simplicity.

The time domain approach is used in this paper. Among many different cycle counting techniques, rain-flow counting is predominantly used and is adopted in the proposed methodology. A detailed description of the rain-flow counting method can be found in [21]. A schematic explanation is shown in Fig. 1 for two different loading histories.

B. Uncertainty Quantification of Material Properties

The prediction of time-dependent fatigue reliability requires uncertainty quantification of the *S–N* curve from constant amplitude loading experiments. Liu and Mahadevan [1] proposed a stochastic *S–N* curve approach to include the autocorrelation between fatigue lives at different stress levels. The fatigue lives N under different constant amplitude tests are treated as random fields/processes with respect to different stress levels s and are assumed to follow the lognormal distribution. The lognormal assumption makes $\log(N(s))$ a Gaussian process with mean value process of $E[\log(N(s))]$ and standard deviation of $\sigma[\log(N(s))]$, where $E[\log(N(s))]$ is the mean *S–N* curve obtained by regression analysis. It has been shown that the variance is not a constant but a function of stress level s [10]. The quantity $\sigma[\log(N(s))]$ represents the scatter in the data and can be obtained by classical statistical analysis. Based on the above assumption, the process

$$Z(s) = \frac{\log(N(s)) - E[\log(N(s))]}{\sigma[\log(N(s))]} \tag{1}$$

is a Gaussian process with zero mean and unit variance.

An exponential decay function is proposed for the covariance function $C(s_1, s_2)$ of $Z(s)$ as

$$C(s_1, s_2) = e^{-\mu|s_1 - s_2|} \tag{2}$$

where μ is a measure of the correlation distance of $Z(s)$ and depends on the material.

Using the Karhunen–Loeve expansion method [22], the fatigue life can be expressed as

$$\log(N(s)) = \sigma[\log(N(s))] \sum_{i=1}^{\infty} \sqrt{\lambda_i}\, \xi_i(\theta) f_i(s) + E[\log(N(s))] \tag{3}$$

where $\sqrt{\lambda_i}$ and $f_i(x)$ are the ith eigenvalues and eigenfunctions of the covariance function $C(s_1, s_2)$. $\xi_i(\theta)$ is a set of independent standard Gaussian random variables.

A detailed explanation and derivation of the stochastic *S–N* curve method can be found in [1]. The major advantage of this method is that it includes the correlation between different stress levels. Most available methods for fatigue reliability analysis assume that the fatigue lives at different stress levels are either uncorrelated $[C(s_1, s_2) = 0]$ random variables or fully correlated $[C(s_1, s_2) = 1]$ variables. These two approaches are named statistical *S–N* curve approach and quantile *S–N* curve approach, respectively, in this discussion. A schematic comparison of the various methods for representing the *S–N* curves is plotted in Fig. 2.

In classical *S–N* fatigue experiments, the specimen is tested until failure or run out at a specified stress level and cannot be tested at the other stress levels. Because of the nonrepeatable nature of fatigue tests, the covariance function cannot be easily observed based on constant amplitude experimental data, which is one possible reason why it has been ignored in the past. However, its effects can be observed under variable amplitude loading. The variation of fatigue lives under variable amplitude loading depends on the variation of fatigue lives at each constant amplitude loading and also their correlations. It has been shown that [1] the two assumptions of covariance (i.e., zero and unity) give upper and lower bounds in the variance prediction under variable amplitude loading. Considering covariance effect leads to a more accurate fatigue life prediction.

C. Damage Accumulation Rule

Liu and Mahadevan [1] proposed the following nonlinear damage accumulation rule based on a modification of Miner's rule. A general form for multiblock loading can be expressed as

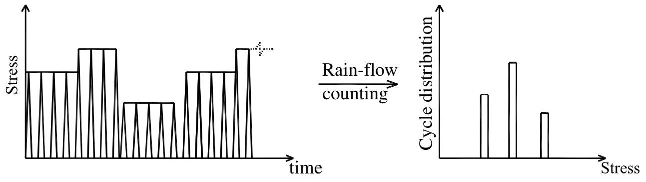
$$\begin{cases} \sum_{i=1}^{k} \frac{n_i}{N_i} = \psi \\ \psi = \sum_{i=1}^{k} \frac{1}{(A_i/\omega_i)+1-A_i} \end{cases} \tag{4}$$

where n_i is the number of applied loading cycles corresponding to the ith load level, and N_i is the number of cycles to failure at the ith load level from constant amplitude experiments. A_i is a material dependent coefficient and ω_i is the cycle distribution at the ith load level from the rain-flow counting results. ψ is a critical damage value, which defines the failure of the material. In Miner's rule, $\psi = 1$ independent of the applied loading. In Eq. (4), ψ depends on the material (A_i) and the applied loading (ω_i). The detailed derivation of the nonlinear damage accumulation rule can be found in [1].

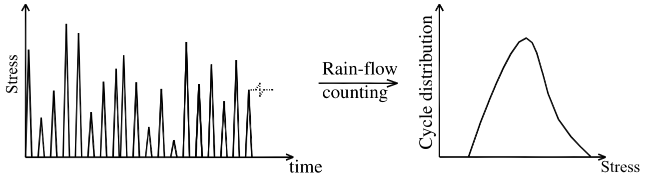
For continuous spectrum loading, Eq. (4) is expressed as

$$\begin{cases} \int \frac{n(s)}{N(s)}\, \mathrm{d}s = \psi \\ \psi = \int \frac{1}{[A(s)/f(s)]+1-A(s)}\, \mathrm{d}s \end{cases} \tag{5}$$

where the cycle distribution ω_i (probability description for block loading) becomes the probability density function (PDF) $f(s)$ of the applied continuous random loading (see Fig. 1).

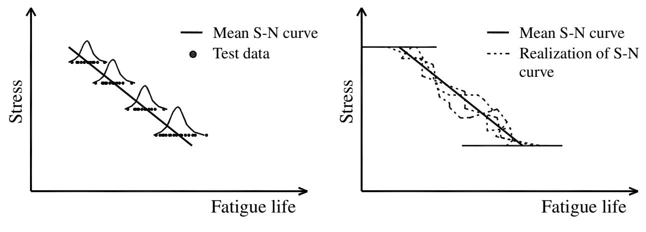


a) Block loading

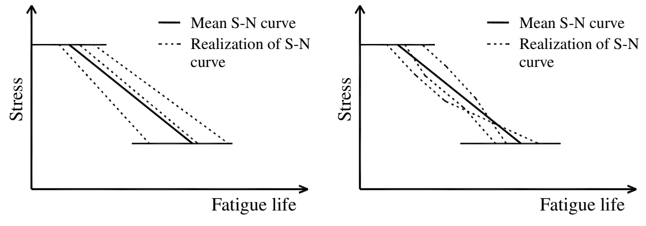


b) Continuous random loading

Fig. 1 Schematic illustration of cycle distribution using rain-flow counting.



a) S-N curve of experiments **b) Statistical S-N curve approach**



c) Quantile S-N curve approach **d) Stochastic S-N curve approach (proposed)**

Fig. 2 Schematic comparisons of different approaches in representing the fatigue *S–N* curve.

When using Eq. (4) [or Eq. (5)] for fatigue life prediction, the critical damage value ψ is first calculated. For repeated multiblock loading, the cycle distribution of the different stress levels at failure can be approximated using the cycle distribution value in a single block. For high-cycle fatigue, this is a reasonable approximation. Then the fatigue life prediction is performed in the same way as the classical procedure using the linear damage rule.

III. Proposed Methods for Time-Dependent Reliability Analysis

Using the uncertainty quantification techniques and damage accumulation rule described in the last section, the reliability can be calculated by numerical simulations, such as the Monte Carlo method. Although the Monte Carlo simulation is powerful in solving the reliability problem, the computational effort prohibits its application. This section proposes two accurate and simple time-dependent reliability calculation methods considering the stochastic damage accumulation. The analysis methods are based on a random process representation of the material *S–N* curve (Sec. II.B) in which the correlation parameter is taken into account. The formulation of the proposed methods is shown next.

A. Fatigue Reliability Under Stationary Loading

First, consider a material under stationary variable amplitude loading. The material *S–N* curve $N(s)$ is described using Eq. (3) as a random process whose covariance function is expressed by Eq. (2). The fatigue damage caused in a single cycle at the stress level s can be expressed as a fraction of the total number of cycles to failure

$$D(s) = \frac{1}{N(s)} \tag{6}$$

Equations (3) and (6) show that the damage in a single cycle can also be expressed as a random process when considering multiple stress levels. The covariance function $C(s_1, s_2)$ of $D(s)$ is assumed to be an exponential decay function as

$$C(s_1, s_2) = \sigma_{D(s_1)}\sigma_{D(s_2)}e^{-\lambda|s_1-s_2|} \tag{7}$$

where $\sigma_{D(s)}$ is the standard deviation of $D(s)$ at the stress level s. λ is a measure of the correlation distance of $D(s)$ and depends on the material.

At any arbitrary time T, the accumulated damage $D_{T,i}$ at the ith stress level s can be expressed as

$$D_{T,i} = \frac{n_i(s)}{N_i(s)} = n_i(s)D_i(S) \tag{8}$$

Under the stationary assumption, the number of applied loading cycles $n_i(s)$ corresponding to the ith load level can be expressed as

$$n_i(s) = Tf_i(s) \tag{9}$$

where $f_i(s)$ is the probability density at the ith load level obtained from the rain-flow counting results. Combining Eqs. (8) and (9) and the damage accumulation rule described in the last section, the total damage at time T considering all the stress levels is the summation of damage at each stress level:

$$D_T = \sum_{i=1}^{\infty} D_{T,i} = \sum_{i=1}^{\infty} \frac{n_i(s)}{N_i(s)} = \sum_{i=1}^{\infty} n_i(s)D_i(S) = T\sum_{i=1}^{\infty} f_i(s)D_i(s) \tag{10}$$

For continuous stationary spectrum loading, Eq. (10) is expressed as

$$D_T = \int_0^{\infty} \frac{n(s)}{N(s)}\,\mathrm{d}s = \int_0^{\infty} n(s)D(s)\,\mathrm{d}s = T\int_0^{\infty} f(s)D(s)\,\mathrm{d}s \tag{11}$$

Equation (11) [or Eq. (10)] is the probabilistic damage growth function of the material under cyclic fatigue loading. It is shown that the damage of the material depends on material properties $D(s)$, applied loading $f(s)$, and time T. The right side of Eq. (11) is an integral of a random process. At a fixed time instant, it becomes a random variable, which is the damage at time T. Under arbitrary external loading, the integral of the random process is not amenable to an analytical solution. Thus, numerical approximation methods are required to calculate the time-dependent reliability.

B. Method 1: Moments Matching Approach

Although the analytical solution of Eq. (11) is not possible, the first two central moments of the fatigue damage can be obtained. For continuous loading, the mean and variance of fatigue damage can be expressed as

$$\begin{cases} \mu_{D_T} = E(T\int_0^{\infty} f(s)D(s)\,\mathrm{d}s) = T\int_0^{\infty} f(s)\mu_{D(s)}\,\mathrm{d}s \\ \sigma^2_{D_T} = COV(T\int_0^{\infty} f(s)D(s)\,\mathrm{d}s) = T^2\int_0^{\infty}\int_0^{\infty} f(s_1)f(s_2)\sigma_{D(s_1)}\sigma_{D(s_2)}e^{-\lambda|s_1-s_2|}\,\mathrm{d}s_1\,\mathrm{d}s_2 \end{cases} \tag{12}$$

For discrete loading, the mean and variance of damage can be expressed as

$$\begin{cases} \mu_{D_T} = T\sum_{i=1}^{\infty} f(s_i)\mu_{D(s_i)} \\ \sigma^2_{D_T} = T^2[\sum_{i=1}^{\infty} f^2(s_i)\sigma^2_{D(s_i)} + 2\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f(s_i)f(s_j)\sigma_{D(s_i)}\sigma_{D(s_j)}e^{-\lambda|s_i-s_j|}] \end{cases} \tag{13}$$

where $\mu_{D(s)}$ and $\sigma_{D(s)}$ in Eqs. (12) and (13) are the mean and standard deviation of damage in a single cycle at stress level s, which are obtained from constant amplitude loading tests.

To calculate the time-dependent fatigue reliability, we need to assume the probability distribution of the fatigue damage D_T because only the first two central moments are available. Equations (10) and (11) can be treated as a summation of a set of random variables. It is well known that a summation of Gaussian random variables is a Gaussian random variable. However, the distribution of the summation of non-Gaussian random variables is usually unknown. Studies for some special cases of summation of non-Gaussian random variables have been reported. Fenton [23] proposed a method to approximate the summation of a set of correlated lognormal random variables as a single lognormal random variable. The method matches the mean and variance of the lognormal sum to the target random variable. It has been shown that this method is very accurate at the tail region, which is usually of the most interest for the reliability analysis. A recent study by Filho and Yacoub [24] showed that the sum of independent identically distributed Weibull variables can also be expressed by a Weibull distribution. The lognormal and Weibull probability distribution functions have been commonly used in the literature to fit the fatigue damage from the constant amplitude loading tests. From the previous discussion, it is shown that the summation of independent and correlated lognormal variables can be approximated by a single lognormal variable using the moment matching method. The summation of independent Weibull variables can be approximated by a single Weibul variable too. The distribution type of the summation of correlated Weibull variables is not available and needs further theoretical study. In the proposed study, we assume that the summation of independent and correlated Weibull variables can be approximated by a single Weibull variable using the moment matching method. We compared this assumption with direct Monte Carlo simulation and found that this assumption can give satisfactory prediction results and may lead to a very small error under certain conditions, as shown in the numerical example in this section.

Once the distribution type of D_T is known or assumed, the reliability can be directly calculated. For example, if D_T follows the lognormal distribution, $\ln(D_T)$ follows the normal distribution with the mean and variance determined by

$$\begin{cases} \bar{\mu}_{D_T} = 2\ln(\mu_{D_T}) - \frac{1}{2}\ln\left(\mu_{D_T}^2 + \sigma_{D_T}^2\right) \\ \bar{\sigma}_{D_T}^2 = -2\ln(\mu_{D_T}) + \ln\left(\mu_{D_T}^2 + \sigma_{D_T}^2\right) \end{cases} \tag{14}$$

The limit state function is defined as shown in Eq. (5). The failure probability P_f is the damage exceedance probability, that is,

$$P_f = P\left(\frac{D_T}{\Psi} \geq 1\right) = \Phi\left(\frac{\bar{\mu}_{D_T} - \ln(\psi)}{\bar{\sigma}_{D_T}}\right) \tag{15}$$

Following the lognormal assumption of the fatigue damage, the time-dependent reliability can be expressed as

$$\text{reliability} = 1 - P_f = 1 - \Phi\left(\frac{\bar{\mu}_{D_T} - \ln(\psi)}{\bar{\sigma}_{D_T}}\right) \tag{16}$$

where Φ is the cumulative density function of the standard Gaussian variable. $\bar{\mu}_{D_T}$ and $\bar{\sigma}_{D_T}$ have been determined by Eqs. (13) and (14). ψ is the critical damage value determined by Eq. (5). Because the variable T is explicitly included in the mean and variance of the fatigue damage, the reliability calculated by Eq. (16) is time dependent. A similar procedure can be followed to calculate the reliability for Weibull distribution.

A numerical example is calculated and compared with direct Monte Carlo simulation to show the accuracy of this moments matching approach. Consider a two-block variable amplitude loading ($S1 = 666$ MPa and $S2 = 478$ MPa). The means of single cycle damage at the two stress levels are $\text{mean}(D(S1)) = 1.89\text{E}-05$ and $\text{mean}(D(S2)) = 2.44\text{E}-06$ using constant amplitude loading for each individual stress level. The standard deviations of single cycle damage at the two stress levels are $\text{Std}(D(S1)) = 3.16\text{E}-06$ and $\text{Std}(D(S2)) = 5.72\text{E}-07$. The Monte Carlo simulation uses 10^6 samples at each time instant and is assumed to be the exact solution. Different factors will affect the moments matching approach: distribution type from constant amplitude test, cycle fraction at each stress level, and the correlation coefficient between single cycle damage at the two stress levels. The cycle fraction effects are compared in Fig. 3 for four different cycle fractions of $S1$ with the correlation coefficient fixed at zero. The correlation effects are compared in Fig. 4 for four different correlation coefficients with the cycle fraction fixed at 0.5. The results of both lognormal and Weibull approximation are plotted and compared together. It is shown that the approximation for the lognormal distribution is very accurate. For Weibull distribution, the results are also very good but may lead to a very small error (see Fig. 3a). Overall, the moments matching method gives very good approximation.

C. Method 2: FORM Approach

The proposed moments matching method needs to assume the type of probability distribution of the accumulated fatigue damage. To calculate the time-dependent reliability without assuming the fatigue damage distribution, another approximation method is proposed based on the FORM. The limit state function $g()$ for the fatigue problem can be expressed based on Eqs. (5) and (10) as

$$g() = \Psi - D_T = \psi - T\sum_{i=1}^{\infty} f_i(s)D_i(s) \tag{17}$$

where $D_1(s), D_2(s), \ldots, D_i(s)$ are a set of correlated random variables which represent the single cycle damage at different stress levels. The surface $g() = 0$, referred to as the limit state, is the boundary between safe and unsafe regions. The failure occurs when $g() < 0$. Therefore, the probability of failure P_f is defined through a multidimensional integral

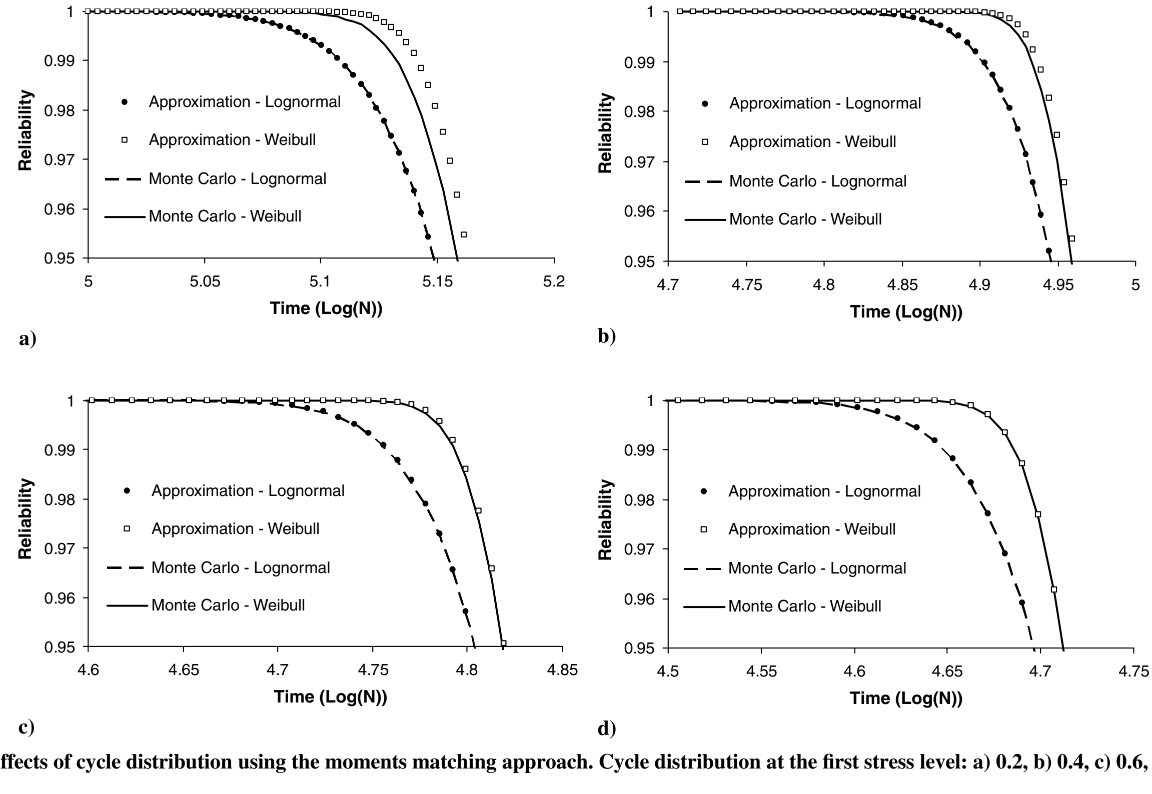


Fig. 3 Effects of cycle distribution using the moments matching approach. Cycle distribution at the first stress level: a) 0.2, b) 0.4, c) 0.6, and d) 0.8.

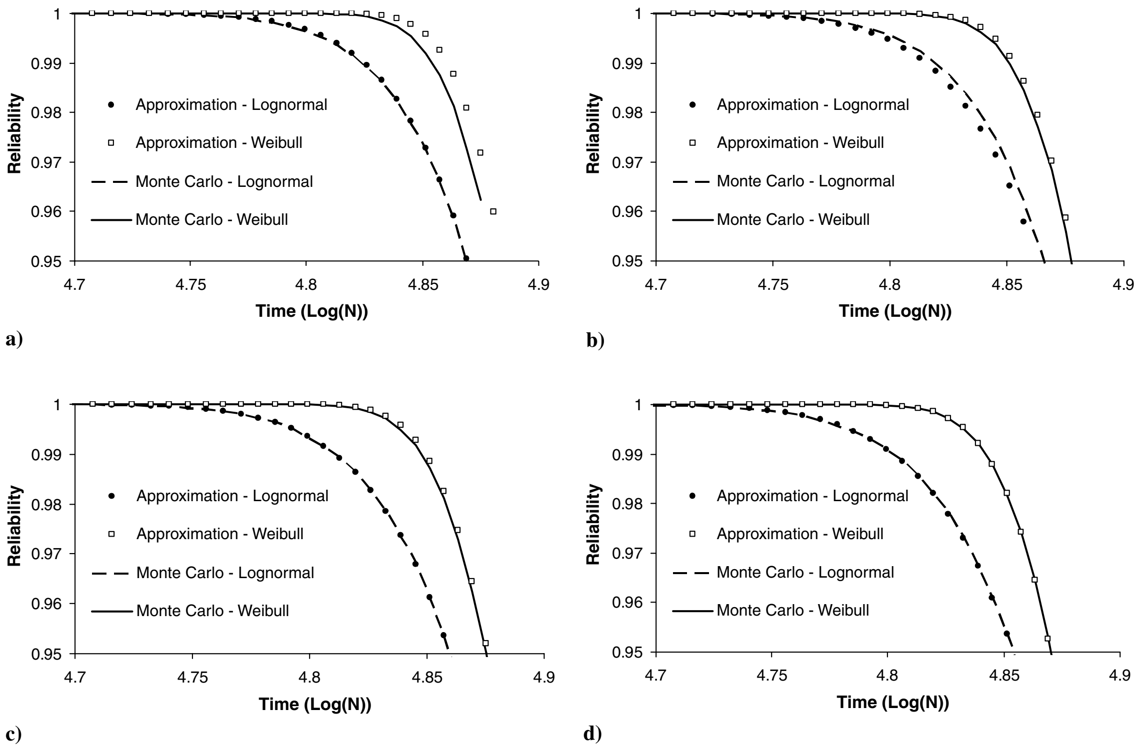


Fig. 4 Effects of correlation using the moments matching approach. Correlation coefficient between the two stress levels: a) 0.0, b) 0.4, c) 0.6, and d) 1.0.

$$P_f = \int \cdots \int_{g()<0} f_D(D_1(s), D_2(s), \ldots, D_i(s))\,\mathrm{d}D_1(s)\,\mathrm{d}D_2(s), \ldots, \mathrm{d}D_i(s) \tag{18}$$

where $f_D(D_1(s), D_2(s), \ldots, D_i(s))$ is the joint probability density function for the basic random variables $D_1(s), D_2(s), \ldots, D_i(s)$ and the integral is performed over the failure region, that is, $g() < 0$.

In general, the multidimensional integral is difficult to evaluate. Various analytical methods have been developed to estimate the value of the integral in Eq. (18). The FORM approach transforms all the random variables to an uncorrelated standard normal space, finds the minimum distance from the limit state to the origin, and estimates the failure probability based on the minimum distance. The minimum distance point on the limit state is also called the most probable point (MPP). The first-order failure probability estimate is computed as

$$P_f = \Phi(-\beta) \tag{19}$$

where β, referred to as the reliability index, is the minimum distance from the origin to the MPP, and Φ is the cumulative distribution function of a standard normal variable. Various techniques can be used to find the MPP. This study uses the recursive formula proposed by Rackwitz and Fiessler [25] to search for the MPP. The FORM method is well established and details can be found in textbooks (e.g., Haldar and Mahadevan [26]).

The limit state function in Eq. (17) includes the variable T and thus is time dependent. At each time instant, the failure probability can be computed using Eq. (19). The same numerical example used in method 1 is also used here to verify the FORM approach. Cycle fraction effects and correlation effects are shown in Figs. 5 and 6, respectively. It is observed that the FORM approach generally gives a very good prediction in all numerical examples for both lognormal and Weibull distributions.

D. Comparison Between Methods 1 and 2

The moments matching method (method 1) assumes the probability distribution of the accumulated fatigue damage D_T under variable amplitude loading and directly calculates the fatigue reliability. The use of this assumption makes the calculation very efficient, which is the major advantage of nethod 1. The disadvantage is also introduced by this assumption. The damage quantity used in fatigue analysis is empirical and is hard to verify by experimental data. As shown in the numerical example, this assumption can lead to some error in the final predictions, although the error appears to not be significant for the example considered.

The FORM method (method 2) does not assume the probability distribution of the accumulated fatigue damage D_T under variable amplitude loading. It only uses the statistics of the basic variables and calculates the joint failure probability. The advantage of the FORM approach is that it is more general and has fewer assumptions. Thus it can accommodate other types of distributions. The computational expense of the FORM approach increases compared to the moments matching approach as the number of random variables increases. The reason is that the FORM computation is iterative. For numerical fatigue reliability calculation under the continuous loading spectra, the external loading can be divided into many small segments, which results in many random variables. Under this condition, it is expected that the FORM approach will not be as efficient as the moments matching approach.

A numerical example is considered later to verify the previous statement. The material properties are the same as the one used in the experimental verification for nickel alloy (see Sec. IV). The cycle distribution of the external loading is assumed to follow the Weibull distribution with a mean value of 600 MPa and a standard deviation of 30 MPa. In the numerical calculation, the continuous cycle distribution is divided into 30 equal segments. The cycle distribution is plotted in Fig. 7a. The prediction results using the moments matching approach, the FORM approach, and the direct Monte Carlo simulation approach are plotted together in Fig. 7b. The results of all three methods are in very close agreement. The computational time for the moments matching approach, FORM approach, and direct Monte Carlo simulation approach are 0.3 s, 1.4 s, and 425 s, respectively.

E. Fatigue Reliability Under Nonstationary Loading

The previous discussion is only applicable to stationary loading because it only considers the cycle distribution of the applied

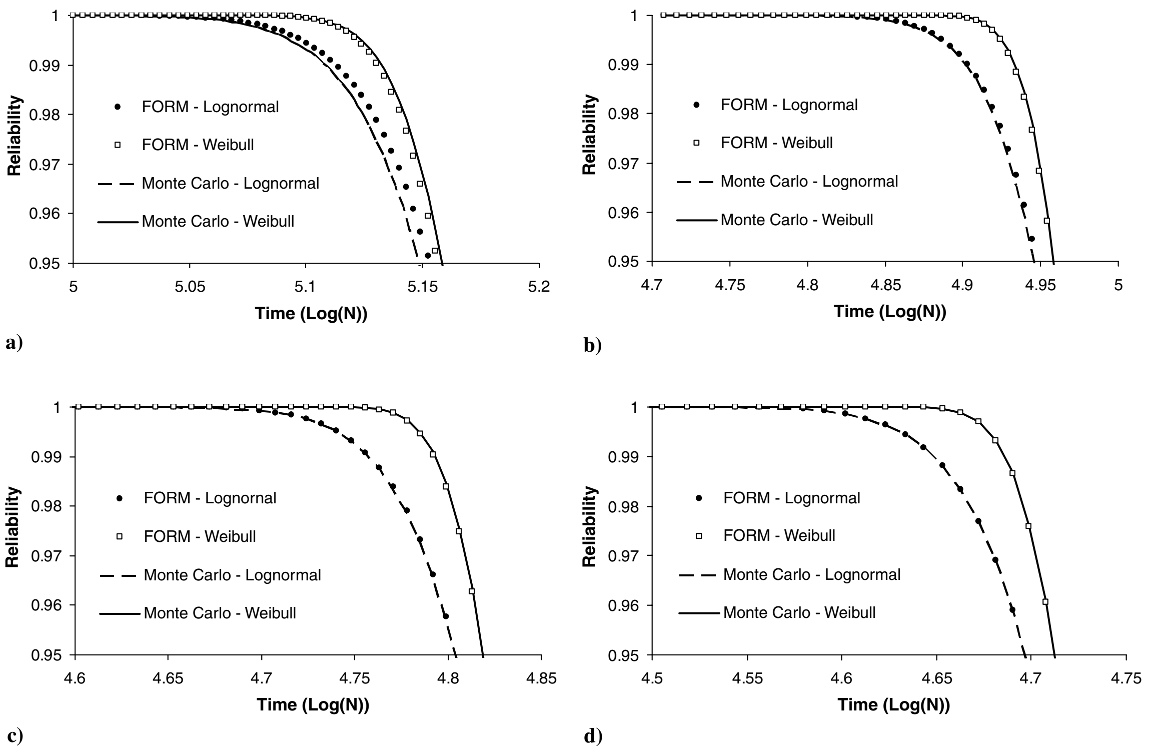


Fig. 5 Effects of cycle distribution using FORM. Cycle distribution at the first stress level: a) 0.2, b) 0.4, c) 0.6, and d) 0.8.

loading. Estimation of fatigue damage accumulation under nonstationary loading is complicated compared to that under stationary loading. For nonstationary applied loading, the cycle distribution changes corresponding to time T. Thus, Eq. (9) is not valid for nonstationary loading. A general expression of the number of applied loading cycles $n_i(s)$ corresponding to the ith load level can be expressed as

$$n_i(s) = Tf_i(s, T) \tag{20}$$

If the nonstationary description of the applied loading [i.e., $f(s, T)$] is known, then the first two central moments of the fatigue damage D_T can be calculated. For continuous loading, the mean value and variance of the fatigue damage can be expressed as

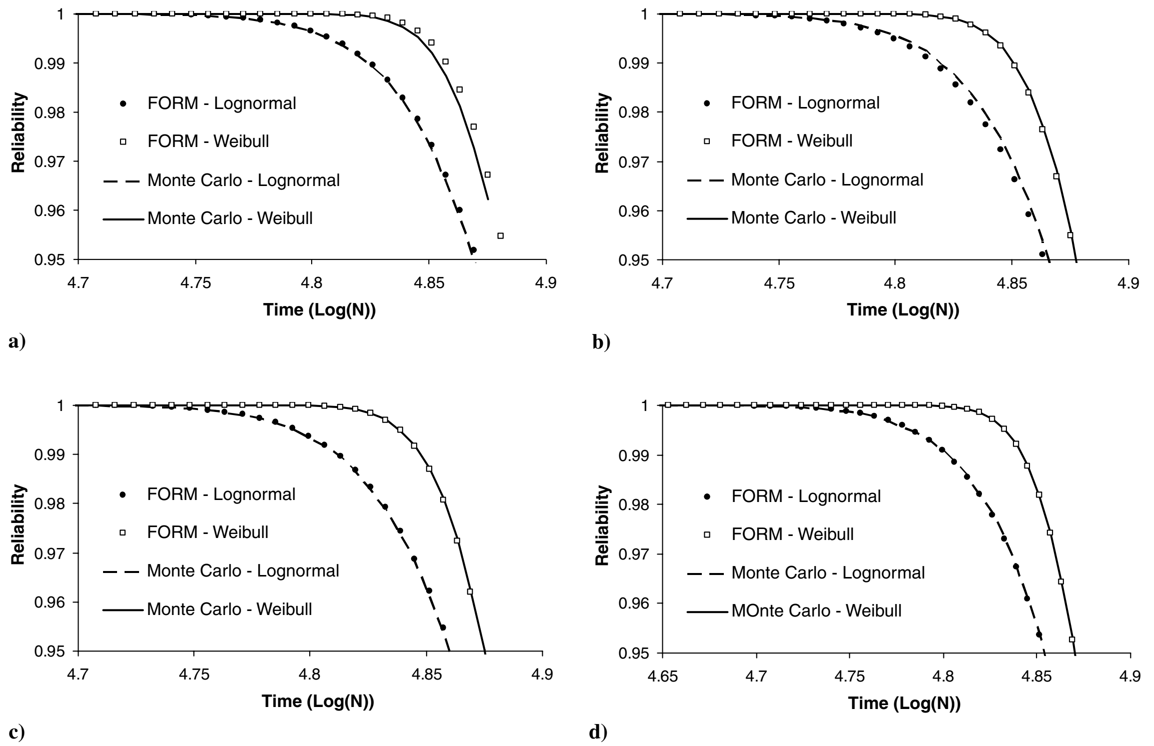


Fig. 6 Effects of correlation using FORM. Correlation coefficient between the two stress levels: a) 0.0, b) 0.4, c) 0.6, and d) 1.0.

$$\begin{cases} \mu_{D_T} = E(T \int_0^\infty f(s,T)D(s)\,\mathrm{d}s) = T \int_0^\infty f(s,T)\mu_{D(s)}\,\mathrm{d}s \\ \sigma^2_{D_T} = COV(T \int_0^\infty f(s,T)D(s)\,\mathrm{d}s) = T^2 \int_0^\infty \int_0^\infty f(s_1,T)f(s_2,T)\sigma_{D(s_1)}\sigma_{D(s_2)}e^{-\lambda|s_1-s_2|}\,\mathrm{d}s_1\,\mathrm{d}s_2 \end{cases} \tag{21}$$

For discrete loading, the mean value and variance of D_T can be expressed as

$$\begin{cases} \mu_{D_T} = T\sum_{i=1}^\infty f(s_i,T)\mu_{D(s_i)} \\ \sigma^2_{D_T} = T^2[\sum_{i=1}^\infty f^2(s_i,T)\sigma^2_{D(s_i)} + 2\sum_{i=1}^\infty \sum_{j=i+1}^\infty f(s_i,T)f(s_j,T)\sigma_{D(s_i)}\sigma_{D(s_j)}e^{-\lambda|s_i-s_j|}] \end{cases} \tag{22}$$

For example, two-step loading is commonly used for variable loading tests under laboratory conditions. The material is first precycled under stress level S_a for T_a cycles. Then the material is cycled till failure at another stress level S_b. This type of loading is nonstationary as the mean and variance of the applied loading vary with time. The cycle distribution function $f(s,T)$ can be expressed for the two-step loading as

$$f(s,T) = \begin{cases} T \le T_a & \begin{cases} 1 & s = S_a \\ 0 & s = S_b \end{cases} \\ T > T_a & \begin{cases} \frac{T_a}{T} & s = S_a \\ \frac{T-T_a}{T} & s = S_b \end{cases} \end{cases} \tag{23}$$

For the moments matching approach, the first two central moments of the fatigue damage D_T can be expressed as

$$\begin{cases} \mu_{D_T} & = \begin{cases} T\mu_{D(s_{ai})} & T \le T_a \\ T_a\mu_{D(s_a)} + (T-T_a)\mu_{D(s_b)} & T > T_a \end{cases} \\ \sigma^2_{D_T} & = \begin{cases} T^2\sigma^2_{D(s_a)} & T \le T_a \\ T_a^2\sigma^2_{D(s_a)} + (T-T_a)^2\sigma^2_{D(s_b)} + 2T_a(T-T_a)\sigma_{D(s_a)}\sigma_{D(s_b)}e^{-\lambda|s_a-s_b|} & T > T_a \end{cases} \end{cases} \tag{24}$$

For the FORM approach, the limit state function can be expressed as

$$g() = \begin{cases} \psi - TD(s_a) & T < T_a \\ \psi - T_aD(s_a) - (T-T_a)D(s_b) & T > T_a \end{cases} \tag{25}$$

The time-dependent fatigue reliability can be calculated following the same procedure described for stationary loading. Equations (24) and (25) show that the reliability variation has two patterns (i.e., before and after T_a). A schematic plot of this phenomenon is shown in Fig. 8.

F. Time-Dependent Fatigue Reliability and Probabilistic Life Distribution

The proposed approximation methods are simple formulations for time-dependent fatigue reliability analysis. Using these methods, the reliability at time T can be calculated. Similarly, for a given reliability level (or probability of failure), the corresponding fatigue life of the material (i.e., time T) can also be calculated. Thus, the current formulation can also be used for probabilistic fatigue life prediction. The probability of fatigue damage being larger than a critical damage amount ψ at time instant T is equal to the probability of fatigue life being less than the time instant T when the fatigue damage is ψ. The relationship of time-dependent failure probability and the probabilistic fatigue life distribution is shown in Fig. 9 schematically. Mathematically, this relation is expressed as

$$P(D_T > \Psi)_{t=T} = P(t < T)_{D_T=\Psi} \tag{26}$$

IV. Experimental Validation

In this section, the prediction results using the proposed methods are compared with experimental data available in the literature. The objective is to examine the applicability of the model to different materials and different loading. The collected experimental data include a wide range of metallic and composite materials under step and multiblock loading. Another guideline in collecting data is that the experimental data should have enough data points both in constant amplitude tests and variable amplitude tests, so that reliable statistical analysis and comparisons can be performed.

A. Experiment Description and Material Fatigue Properties

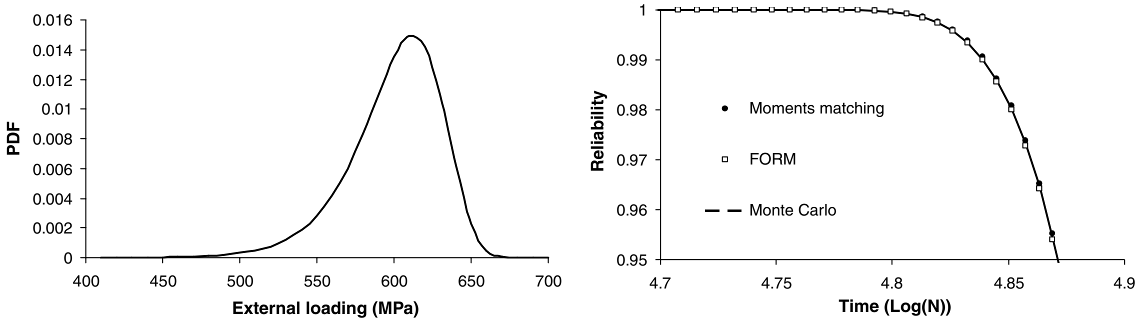
A brief summary of the collected experimental data is shown in Table 1, which includes material name, reference, variable loading type, and specimen numbers at constant and variable loading tests. Figure 10 provides schematic illustrations of the variable loading type listed in Table 1.

The statistics of the experimental data under constant amplitude loading are shown in Table 2, including mean value, standard deviation, and distribution type of the single cycle fatigue damage at different stress levels.

B. Validation of the Reliability Estimation

The final objective of time-dependent fatigue reliability is to predict the reliability variation corresponding to time under different variable loading. In this section, the predicted reliability variation is compared with the empirical fatigue reliability variation from the experimental data. Because of the large number of experimental data collected in this study and the space limitations, we only show the comparisons under several loading conditions for each material. The comparisons are shown in Fig. 11 by plotting the predicted and experimental variations together. The details of the plotted experimental loading conditions are listed in Table 3.

It is observed that the prediction results agree with the experimental results very well for different variable amplitude loading, with a few exceptions. The prediction results shown in Fig. 11 are obtained using the FORM approach. Although it is not



a) Cycle distribution **b) Comparison with the Monte Carlo simulation**

Fig. 7 Comparison between the moments matching approach and the FORM approach for continuous loading.

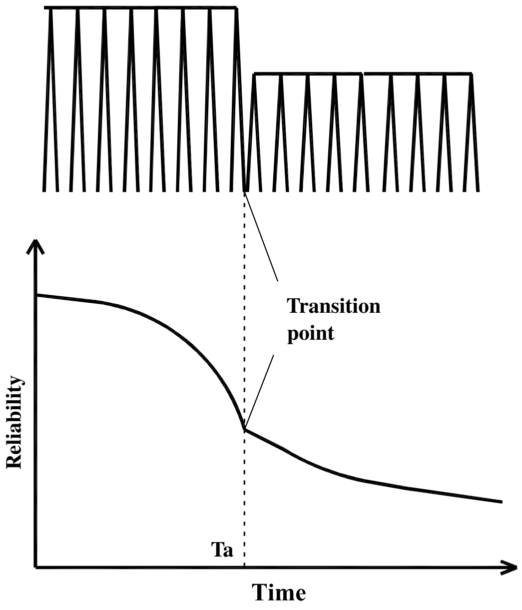


Fig. 8 Schematic reliability variation under step loading.

shown here, the moments matching approach yields very similar prediction results compared to the FORM approach. Because only block and step loading data are used here, the computational time of both the moments matching approach and the FORM approach are almost identical.

V. Conclusions

Two efficient fatigue reliability calculation methods are proposed in this study. They are based on a stochastic process representation of the material properties under constant amplitude loading and a

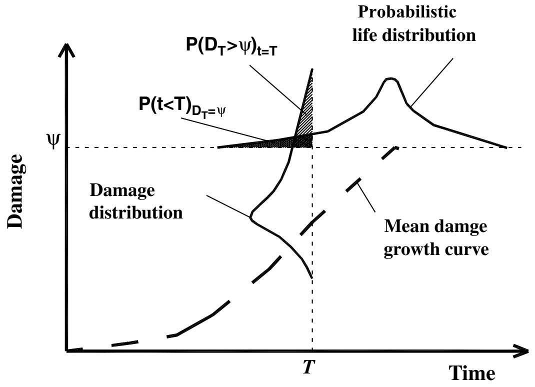
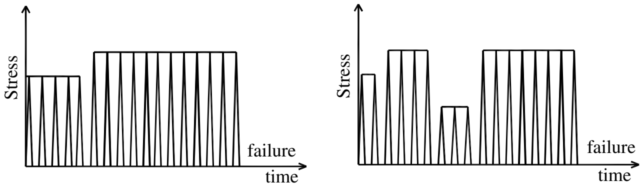
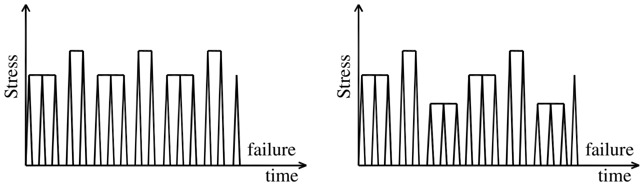


Fig. 9 Schematic illustration of probabilistic fatigue life distribution and time-dependent fatigue reliability.



a) TS (two-step loading) **b) MS (multistep loading)**



c) TB (two-block loading) **d) MB (multiblock loading)**

Fig. 10 Illustration of the type of variable loading used in this study.

Table 1 Experimental description of collected materials

			No. of specimens[b]	
Material name	Reference	Types of variable loading[a]	Constant loading	Variable loading
Nickel–silver	[27]	TS	200	50
16Mn steel	[28]	TS and MS	15	10
LY12CZ aluminum alloy	[29]	MB	N/A	15–21
Carbon steel	[28]	TS	15–18	13–15
45 steel-1	[13]	MB	10	9
45 steel-2	[30]	MB	10	6
DD16 fiberglass composite laminates	[31]	TB	15–20	3–62

[a] The abbreviation and schematic illustration of the type of the variable loading is shown in Fig. 4.

[b] The number of specimens indicates the number under the same stress level (constant loading) or the same type of variable loading.

Table 2 Statistics of constant amplitude *S–N* curve data

Material	Stress amplitude, MPa	Statistics of single cycle fatigue damage $(1/N)$		
		Mean	Std.	Distribution
Nickel–silver	478	2.44E – 06	5.72E – 07	Weibull
	583	8.32E – 06	1.79E – 06	Weibull
	666	1.89E – 05	3.16E – 06	Weibull
16Mn steel	394	9.21E – 06	2.14E – 06	Weibull
	373	5.16E – 06	6.48E – 07	Weibull
	344	1.42E – 06	2.42E – 07	Weibull
LY12CZ	125.44	4.37E – 05	1.05E – 05	Weibull
	101.92	1.76E – 05	1.77E – 06	Weibull
	78.79	7.05E – 06	1.43E – 06	Weibull
	49.98	2.37E – 06	8.33E – 07	Weibull
	46.06	1.15E – 06	7.19E – 07	Weibull
	37.04	1.57E – 07	4.83E – 08	Weibull
Carbon steel	366	2.05E – 05	7.46E – 06	Weibull
	331	7.27E – 06	2.43E – 06	Weibull
	309	1.72E – 06	7.33E – 07	Weibull
45 steel-1	525	5.88E – 06	3.94E – 06	Lognormal
	500	4.18E – 06	3.09E – 06	Lognormal
	475	3.08E – 06	1.8E – 06	Lognormal
	450	1.99E – 06	1.44E – 06	Lognormal
	400	8.23E – 07	4.86E – 07	Lognormal
45 steel-2	750	3.38E – 05	1.03E – 05	Lognormal
	650	1.05E – 05	3.85E – 06	Lognormal
	630	9.54E – 06	2.51E – 06	Lognormal
	590	5.82E – 06	1.21E – 06	Lognormal
	520	2.55E – 06	1.38E – 06	Lognormal
DD16 fiberglass composite laminates	206	5.48E – 06	7.17E – 06	Lognormal
	241	1.97E – 05	1.97E – 05	Lognormal
	328	0.000615	0.000362	Lognormal
	414	0.004569	0.003278	Lognormal

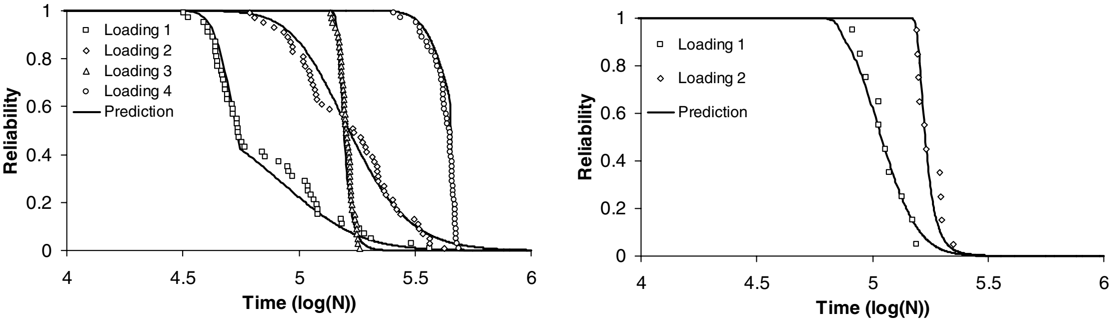
nonlinear damage accumulation rule. In the moments matching approach, the fatigue damage under variable amplitude loading is assumed to follow either lognormal or Weibull distribution, whereas the first two central moments are determined analytically without approximation. This results in a simple analytical solution for either the probability distribution of the service time to failure (fatigue life) or the probability distribution of the amount of damage at any service time. In the FORM approach, no assumption is made for the damage distribution under variable amplitude loading and the statistics of the basic variables is used together with the first-order reliability method.

The proposed methods are very efficient in calculating the time-dependent reliability variation under cyclic fatigue loading compared to the simulation-based approaches. Thus, the proposed methods are appropriate for application for preliminary analysis at the design stage. The other advantage of the proposed method is that they include the correlation effect of the damage accumulation under variable amplitude loading, which has been mostly ignored in the existing models. Currently available models in the literature are shown to be two special cases of the proposed approach, that is, independent random variables and fully correlated random variables. The proposed methodology has been validated using experimental data under deterministic variable amplitude loading. Further validation and modification are required to consider other types of uncertainties associated with external loading, such as uncertainty due to insufficient data, modeling uncertainty, etc. Application of the proposed methods to structural systems and inclusion of uncertainties in structural geometry and operational conditions also needs further study.

Table 3 Experiments description shown in Fig. 11

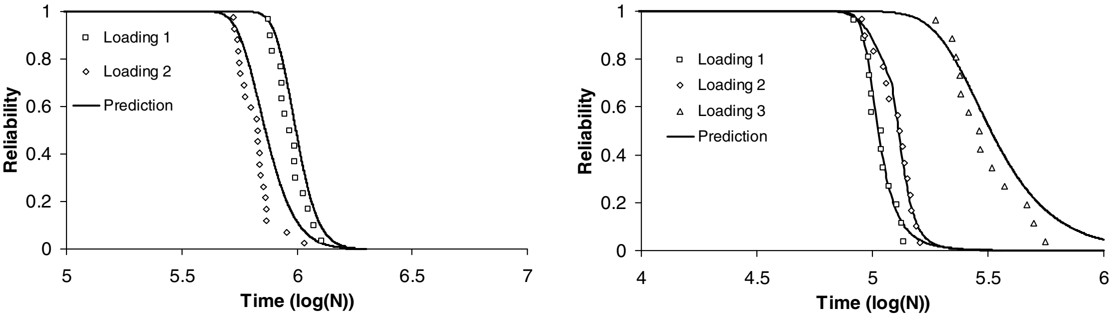
Material	Loading no.	Variable loading[a]
Nickel–silver	1	TS: 666 $(5.54 \times 10^4) \rightarrow 478$ (X)
	2	TS: 666 $(3.98 \times 10^4) \rightarrow 478$ (X)
	3	TS: 478 $(1.15 \times 10^5) \rightarrow 666$ (X)
	4	TS: 478 $(4.46 \times 10^5) \rightarrow 666$ (X)
16Mn steel	1	TS: 394 $(7.5 \times 10^4) \rightarrow 373$ (X)
	2	TS: 373 $(1.46 \times 10^5) \rightarrow 394$ (X)
	3	MS: 373 $(10^5) \rightarrow 394$ $(10^5) \rightarrow 373$ $(10^5) \rightarrow 344$ $(10^5) \rightarrow 394$ $(10^5) \rightarrow 344$ $(10^5) \rightarrow 394$ (X)
LY12CZ	1	MB: 93.1 $(2.64 \times 10^3) \rightarrow 69.58$ $(1.056 \times 10^4) \rightarrow 46.06$ $(1.848 \times 10^4) \rightarrow 23.52$ (3.432×10^4)
	2	MB: 93.1 $(6.6 \times 10^2) \rightarrow 69.58$ $(3.3 \times 10^3) \rightarrow 55.86$ $(6.6 \times 10^3) \rightarrow 46.06$ $(1.584 \times 10^4) \rightarrow 37.24$ (3.96×10^4)
Carbon steel	1	TS: 331 $(8.06 \times 10^4) \rightarrow 366$ (X)
	2	TS: 331 $(1.21 \times 10^5) \rightarrow 366$ (X)
	3	TS: 331 $(8.06 \times 10^5) \rightarrow 309$ (X)
45 steel-1	1	MB: 240 $(10^5) \rightarrow 350$ $(8 \times 10^4) \rightarrow 400$ $(2.5 \times 10^4) \rightarrow 500$ $(10^4) \rightarrow 400$ $(2.5 \times 10^4) \rightarrow 350$ $(8 \times 10^4) \rightarrow 240$ (10^5)
45 steel-2	2	MB: 500 $(1.5 \times 10^4) \rightarrow 590$ $(4 \times 10^3) \rightarrow 626.6$ $(5 \times 10^3) \rightarrow 590$ $(4 \times 10^3) \rightarrow 500$ (1.5×10^4)
DD16	3	TB: 328 $(10) \rightarrow 207$ (10^3)

[a] The number before the parentheses indicates the stress level and the number inside the parentheses is the applied number of cycles. For step loading (TS and MS), the applied cycle number of the last stress level is not known a priori and thus an X is used.



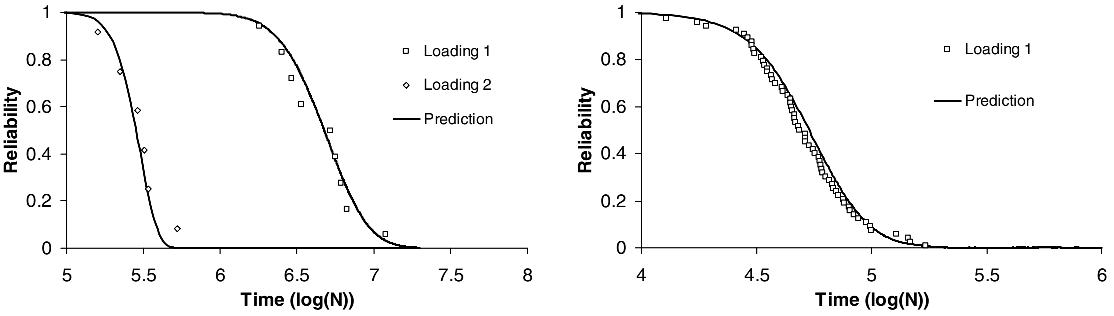
a) Nickel-silver

b) 16Mn steel



c) LY12CZ aluminum alloy

d) Carbon steel



e) 45 steel

f) DD16 composite

Fig. 11 Time-dependent reliability variation comparisons between prediction and experimental results.

Acknowledgment

The research reported in this paper was supported by funds from the Federal Aviation Administration William J. Hughes Technical Center (Contract No. DTFACT-06-C-00017, Project Manager Dy Le). The support is gratefully acknowledged.

References

[1] Liu Y., and Mahadevan, S., "Stochastic Fatigue Damage Modeling Under Variable Amplitude Loading," *International Journal of Fatigue*, Vol. 29, 2007, pp. 1149–1161. doi:10.1016/j.ijfatigue.2006.09.009

[2] Liao, M., Xu, X., and Yang, Q. X., "Cumulative Fatigue Damage Dynamic Interference Statistical Model," *International Journal of Fatigue*, Vol. 17, 1995, pp. 559–566. doi:10.1016/0142-1123(96)81216-3

[3] Kam, T. Y., Chu, K. H., and Tsai, S. Y., "Fatigue Reliability Evaluation for Composite Laminates via a Direct Numerical Integration Technique," *International Journal of Solids and Structures*, Vol. 35, 1998, pp. 1411–1423. doi:10.1016/S0020-7683(97)00114-5

[4] Le, X., and Peterson, M. L., "A Method for Fatigue Based Reliability When the Loading of a Component Is Unknown," *International Journal of Fatigue*, Vol. 21, 1999, pp. 603–610. doi:10.1016/S0142-1123(99)00016-X

[5] Shen, H., Lin, J., and Mu, E., "Probabilistic Model on Stochastic Fatigue Damage," *International Journal of Fatigue*, Vol. 22, No. 7, 2000, pp. 569–572.

[6] Kaminski, M., "On Probabilistic Fatigue Models for Composite Materials," *International Journal of Fatigue*, Vol. 24, 2002, pp. 477–495. doi:10.1016/S0142-1123(01)00103-7

[7] Shimokawa, T., and Tanaka, S., "A Statistical Consideration of Miner's Rule," *International Journal of Fatigue*, Vol. 2, No. 4, 1980, pp. 165–170. doi:10.1016/0142-1123(80)90044-4

[8] Kopnov, V. A., "A Randomized Endurance Limit in Fatigue Damage Accumulation Models," *Fatigue and Fracture of Engineering Materials and Structures*, Vol. 16, 1993, pp. 1041–1059. doi:10.1111/j.1460-2695.1993.tb00077.x

[9] Kopnov, V. A., "Intrinsic Fatigue Curves Applied to Damage Evaluation and Life Prediction of Laminate and Fabric Material," *Theoretical and Applied Fracture Mechanics*, Vol. 26, 1997, pp. 169–176. doi:10.1016/S0167-8442(96)00046-8

[10] Pascual, F. G., and Meeker, W. Q., "Estimating Fatigue Curves with the Random Fatigue-Limit Model," *Technometrics*, Vol. 41, 1999, pp. 277–302. doi:10.2307/1271342

[11] Rowatt, J. D., and Spanos, P. D., "Markov Chain Models for Life Prediction of Composite Laminates," *Structural Safety*, Vol. 20, 1998, pp. 117–135. doi:10.1016/S0167-4730(97)00025-8

[12] Ni, K., and Zhang, S., "Fatigue Reliability Analysis Under Two-Stage Loading," *Reliability Engineering and System Safety*, Vol. 68, No. 2,

2000, pp. 153–158.

[13] Zheng, X., and Wei, J., "On the Prediction of P–S–N Curves of 45 Steel Notched Elements and Probability Distribution of Fatigue Life Under Variable Amplitude Loading from Tensile Properties," *International Journal of Fatigue*, Vol. 27, 2005, pp. 601–609. doi:10.1016/j.ijfatigue.2005.01.001

[14] Manson. S. S., and Halford, G. R., "Practical Implementation of the Double Linear Damage Rule and Damage Curve Approach for Treating Cumulative Fatigue Damage," *International Journal of Fracture*, Vol. 17, 1981, pp. 169–192. doi:10.1007/BF00053519

[15] Halford, G. R., "Cumulative Fatigue Damage Modeling—Crack Nucleation and Early Growth," *International Journal of Fatigue*, Vol. 19, 1997, pp. 253–260. doi:10.1016/S0142-1123(97)00048-0

[16] Jiao, G., "A Theoretical Model for the Prediction of Fatigue Under Combined Gaussian and Impact Loads," *International Journal of Fatigue*, Vol. 17, 1995, pp. 215–219. doi:10.1016/0142-1123(95)98942-V

[17] Tovo, R., "A Damage-Based Evaluation of Probability Density Distribution for Rain-Flow Ranges from Random Processes," *International Journal of Fatigue*, Vol. 22, 2000, pp. 425–429. doi:10.1016/S0142-1123(00)00006-2

[18] Fu, T. T., and Cebon, D., "Predicting Fatigue Lives for Bi-Modal Stress Spectral Densities," *International Journal of Fatigue*, Vol. 22, 2000, pp. 11–21. doi:10.1016/S0142-1123(99)00113-9

[19] Banvillet, A., Lagoda, T., Macha, E., Niesony, A., Palin-Luc, T., and Vittori, J. F., "Fatigue Life Under Non-Gaussian Random Loading from Various Models," *International Journal of Fatigue*, Vol. 26, 2004, pp. 349–363. doi:10.1016/j.ijfatigue.2003.08.017

[20] Benasciutti, D., and Tovo, R., "Spectral Methods for Lifetime Prediction Under Wide-Band Stationary Random Processes," *International Journal of Fatigue*, Vol. 27, 2005, pp. 867–877. doi:10.1016/j.ijfatigue.2004.10.007

[21] Suresh, S., *Fatigue of Materials*, 2nd ed., Cambridge University Press, Cambridge, England, U.K., 1998, pp. 265–267.

[22] Loeve, M., *Probability Theory*, 4th ed., Springer, New York, 1977.

[23] Fenton, L. F., "The Sum of Lognormal Probability Distributions in Scatter Transmission Systems," *IRE Transactions on Communications Systems*, Vol. 8, 1960, pp. 57–67. doi:10.1109/TCOM.1960.1097606

[24] Filho, J., and Yacoub, M., "Simple Precise Approximations to Weibull Sums," *IEEE Communications Letters*, Vol. 10, No. 8, 2006, pp. 614–616. doi:10.1109/LCOMM.2006.1665128

[25] Rackwitz, R., and Fiessler, B., "Structural Reliability Under Combined Random Load Sequences," *Computer and Structures*, Vol. 9, No. 5, 1978, pp. 484–494.

[26] Haldar, A., and Mahadevan, S., *Probability, Reliability, and Statistical Methods in Engineering Design*, Wiley, New York, 2000.

[27] Tanaka, S., Ichikawa, M., and Akita, S., "A Probabilistic Investigation of Fatigue Life and Cumulative Cycle Ratio," *Engineering Fracture Mechanics*, Vol. 20, 1984, pp. 501–513. doi:10.1016/0013-7944(84)90055-9

[28] Xie, L., "Equivalent Life Distribution and Fatigue Failure Probability Prediction," *International Journal of Pressure Vessels and Piping*, Vol. 76, 1999, pp. 267–273. doi:10.1016/S0308-0161(98)00117-3

[29] Wu, Y., "Experimental Verification and Statistical Analysis of Miner's Accumulative Damage Rule," *Chinese Journal of Aeronautics*, Vol. 6, 1985, pp. 351–360 (in Chinese).

[30] Yan, J. H., Zheng, X. L., and Zhao, K., "Prediction of Fatigue Life and its Probability Distribution of Notched Friction Welded Joints Under Variable-Amplitude Loading," *International Journal of Fatigue*, Vol. 22, 2000, pp. 481–494. doi:10.1016/S0142-1123(00)00021-9

[31] Mandell, J. F., and Samborsky, D. D., "DOE/MSU Composite Materials Fatigue Database: Test Methods, Materials, and Analysis," Sandia National Laboratories, Albuquerque, NM, Feb. 2003.

A. Roy
Associate Editor